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A SEMI-INFINITE CRACK IN FRONT OF A CIRCULAR, THERMALLY MISMATCHED HETEROGENEITY

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Abstract—The stress intensity factors (SIFs) of a semi-infinite crack in front of a thermally and elastically mismatched, circular heterogeneity are studied based on a singular integral equation technique and on a self-consistent method. It is shown that the solution resulting from the self-consistent method is equivalent to the one from the Cauchy-type singular integral equation if the kernel function in the integral equation is completely ignored. The self-consistent solution is then compared with the numerical solution of the integral equation for the full range of elastic mismatch using various discretization techniques. For Dundurs' parameters within the range $|\alpha| \leqslant 0.6$ and $\beta = \alpha/4$, the SIFs predicted by the self-consistent formula agree within 7% or better when compared with the numerical results, provided that the crack tip is not situated extremely close to the heterogeneity.

Finally, it is analyzed how the convergence of the SIFs of crack tips which are extremely close to the heterogeneity is influenced by the choice of discretization scheme: to generate computer codes which are easy to implement, time-efficient and numerically accurate, it is advantageous to use techniques which operate on a finite interval $[-1, +1]$ (i.e. Gauss–Chebyshev, Lobatto–Chebyshev) as compared with those which cover the positive x-axis $[0, \infty)$ (i.e. Radau–Chebyshev or Gauss–Hermite). Consequently, it is advisable to map the semi-infinite crack into a crack of finite size by using suitable transforms. It will be shown that among the discretizations for a finite interval the fastest to converge are those which explicitly use the end points -1 and $+1$ (Lobatto–Chebyshev) followed closely by polynomial extrapolation of discrete solution data for $(-1, +1)$ into the crack tips.

1. FORMULATION OF THE PROBLEM

Consider the plane elastic problem shown in Fig. 1. An elastic matrix contains a single heterogeneity, i.e. a fiber or inclusion, of radius R, together with a semi-infinite crack in the radial direction, in the mid-plane of the fiber, at a distance ε. The elastic constants of the matrix and of the fiber are denoted by (μ_1, κ_1) and (μ_2, κ_2), respectively, where μ_i is the shear modulus and $\kappa_i = 3-4\nu_i$ (plane strain) or $\kappa_i = (3-\nu_i)/(1+\nu_i)$ (plane stress) is Muskhelishvili's constant, ν_i being Poisson's ratio and $i = 1, 2$. The corresponding thermal expansion coefficients of the matrix and of the fiber material are denoted by α_1^* and α_2^*, respectively.

The goal of this paper is to study the influence of the elastic and thermal mismatch on the stress concentration, K_I, of the semi-infinite crack. To this end an analytical formula for the stress intensity factor (SIF) will be presented, with the assumption of small elastic mismatch. This formula follows either from a self-consistent method, which was used by

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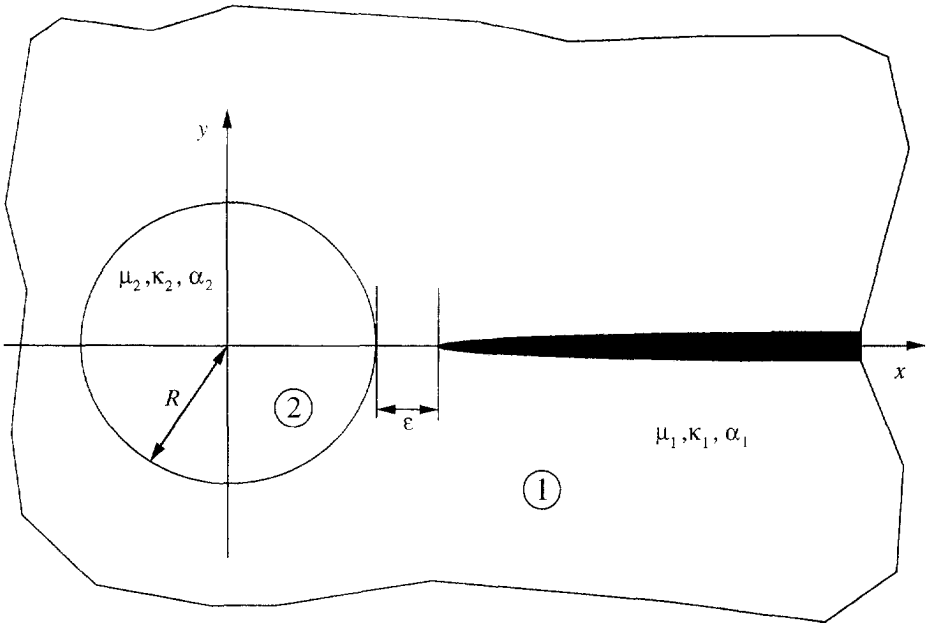


Fig. 1. Geometry of the semi-infinite crack in front of an inclusion.

Kachanov *et al.* (1990), Rodin and Yuh-Long Hwang (1991) and Gong and Meguid (1992), or by applying an inversion formula to a Cauchy-type singular integral equation (CSIE) using results from Section 90 of Muskhelishvili's book on singular integral equations (1992).

In addition to the analytical result the general CSIE will be solved numerically using standard discretization techniques developed by Erdogan and co-workers [see Erdogan *et al.* (1973); Ming-Che Lu and Erdogan (1983)] as well as Ioakimidis (1976) and Ioakimidis and Theocaris (1977, 1980). However, if the crack gets very close to the inclusion convergence will become an issue. It will then be necessary to apply special numerical measures, e.g. polynomial extrapolation or the Lobatto Chebyshev scheme as proposed by Theocaris and Ioakimidis (1977).

By comparison of both predictions for K_I, the range of validity of the analytical formula can finally be assessed.

2. THE INTEGRAL EQUATION

Based on the original work by Erdogan *et al.* (1973, 1974) and Erdogan and Gupta (1975) for cracks of finite size, and referring to an application of this work by Müller and Schmauder (1993), the CSIE for the semi-infinite crack shown in Fig. 1 can be written as follows:

$$\int_{R+\varepsilon}^{\infty} \frac{f(t)}{t-x}\,dt + \int_{R+\varepsilon}^{\infty} k(x,t)f(t)\,dt = -\frac{\pi(1+\kappa_1)}{2\mu_1}p(x), \quad R+\varepsilon < x < \infty, \tag{1}$$

where $f(t)$ denotes an unknown distribution of dislocations which is used to simulate the crack. The coordinates x and t characterize arbitrary points on the crack flanks. Mechanical and thermal loads acting in the uncracked matrix at points x of the prospective crack are contained in the expression $p(x)$. In particular, for thermal loads resulting from thermal mismatch between the fiber and the matrix it can be shown that (Müller and Schmauder, 1993):

$$p(x) = \frac{2\mu_1}{1+\kappa_1}\Delta\alpha^*\Delta T\frac{2(1+\alpha)}{1+\alpha-2\beta}\frac{R^2}{x^2} \tag{2}$$

with

$$\Delta\alpha^*\Delta T = \begin{cases} [(1+\nu_2)\alpha_2^* - (1+\nu_1)\alpha_1^*](T-T_R), & \text{plane strain} \\ (\alpha_2^* - \alpha_1^*)(T-T_R), & \text{plane stress,} \end{cases} \tag{3}$$

T and T_R being the current and the reference temperature of the material, respectively.

The contraction $k(x,t)$ denotes an integral kernel which can be separated into a singular and into a regular part (Erdogan and Gupta, 1975):

$$k(x,t) = k_s(x,t) + k_r(x,t), \tag{4}$$

where

$$k_s(x,t) = \frac{1}{t-s}\left\{-\frac{\alpha+\beta^2 s}{1-\beta^2 x} + \frac{\beta-\alpha}{1+\beta}\frac{1}{x^2}(3s^2-R^2)\left(1-\frac{2s}{t}\right)\right\}$$
$$+\frac{\beta-\alpha}{1+\beta}\left\{\left(1-\frac{4s}{t}\right)\frac{s(s^2-R^2)}{x^2(t-s)^2} - \frac{s^3(s^2-R^2)^2}{R^4 t(t-s)^3}\right\} \tag{5}$$

and

$$k_r(x,t) = \frac{\beta-\alpha}{1+\beta}\frac{R^2}{x^2}\left(\frac{1}{2t}-\frac{3R^2}{tx^2}\right) - \left(\frac{1-\alpha^2}{(1-\beta)(1+\alpha-2\beta)}-1\right)\frac{R^2}{2tx^2} \tag{6}$$

with

$$s = \frac{R^2}{x^2}, \quad m = \frac{\mu_2}{\mu_1}$$
$$\alpha = \frac{m(\kappa_1+1)-(\kappa_2+1)}{m(\kappa_1+1)+(\kappa_2+1)}, \quad \beta = \frac{m(\kappa_1-1)-(\kappa_2-1)}{m(\kappa_1-1)+(\kappa_2+1)}, \tag{7}$$

where the Dundurs' parameters, α and β, have been introduced. Note that $k_s(x,t)$ becomes singular if the crack tip ends at the fiber matrix interface.

In particular, for equal elastic but different thermal expansion coefficients,† eqns (1), (2) and (4)–(7) can be combined and result in the following simple relation:

$$\int_{R+\varepsilon}^{\infty} \frac{f(t)}{t-x}\,dt = -2\pi[\alpha_2^* - \alpha_1^*](T-T_R)\frac{R^2}{x}, \quad R+\varepsilon < x < \infty. \tag{8}$$

In order to solve the CSIEs (1) or (8) it is necessary to transform them into dimensionless form, e.g. by using the following formulae:

$$t = \varepsilon + R(1+\bar{t}), \quad x = \varepsilon + R(1+\bar{x}), \quad \bar{t}\in[0,\infty), \quad \bar{x}\in(0,\infty) \tag{9}$$

which inserted into eqns (1) and (2) lead to:

$$\int_0^{\infty}\frac{f(\bar{t})}{\bar{t}-\bar{x}}\,d\bar{t} + \int_0^{\infty} Rk(\bar{x},\bar{t})f(\bar{t})\,d\bar{t} = -\frac{\pi(1+\kappa_1)}{2\mu_1}p(\bar{x}) \tag{10}$$

with

† The elastically homogeneous case follows if α and β are equal to zero.

$$p(\bar{x}) = \frac{2\mu_1}{1+\kappa_1}\,\Delta\alpha^*\Delta T\,\frac{2(1+\alpha)}{1+\alpha-2\beta}\,\frac{1}{[1+\bar{x}+\varepsilon/R]^2}. \tag{11}$$

Next, the distribution of dislocations, $f(t)$, is separated into a singular and into a regular part, $F(t)$, as follows:

$$f(t) = F(t)[t-(R+\varepsilon)]^{-1/2} = \frac{F(\bar{t})}{\sqrt{(R\bar{t})}}. \tag{12}$$

Introducing

$$h(\bar{t}) = \frac{F(\bar{t})}{\sqrt{R}\Delta\alpha^*\Delta T}, \quad w(\bar{t}) = \frac{1}{\sqrt{t}} \tag{13}$$

the CSIE (10) can also be expressed as:

$$\int_0^1 \frac{h(\bar{t})w(\bar{t})}{\bar{t}-\bar{x}}\,\mathrm{d}\bar{t} + \int_0^1 Rk(\bar{x},\bar{t})h(\bar{t})w(\bar{t})\,\mathrm{d}\bar{t} = -\frac{2\pi(1+\alpha)}{1+\alpha-2\beta}\,\frac{1}{[1+\bar{x}+\varepsilon/R]^2}. \tag{14}$$

For numerical reasons it is advantageous, although not imperative, to eliminate the infinite boundary in the integrals. This can also be achieved by suitable coordinate transformation. The following transformation formulae stem from a more recent paper by Hutchinson *et al.* (1987). They allow one to map the interval $[R+\varepsilon, \infty)$ directly onto the dimensionless set $[-1, +1]$:

$$t = \varepsilon + R\frac{2}{1-\bar{t}}, \quad x = \varepsilon + R\frac{2}{1-\bar{x}}, \quad \bar{t},\bar{x}\in[-1,+1] \tag{15}$$

which, inserted into eqns (1) and (2), result in:

$$\int_{-1}^{+1} \frac{f(\bar{t})}{1-t}\,\frac{1}{\bar{t}-\bar{x}}\,\mathrm{d}\bar{t} + \int_{-1}^{+1} \frac{2Rk(\bar{x},\bar{t})}{(1-\bar{x})(1-\bar{t})}\,\frac{f(\bar{t})}{1-\bar{t}}\,\mathrm{d}\bar{t} = -\frac{\pi(1+\kappa_1)}{2\mu_1}\,\frac{p(\bar{x})}{1-\bar{x}}, \tag{16}$$

where

$$p(\bar{x}) = \frac{2\mu_1}{1+\kappa_1}\,\Delta\alpha^*\Delta T\,\frac{2(1+\alpha)}{1+\alpha-2\beta}\,\frac{(1-\bar{x})^2}{[2+\varepsilon/R(1-\bar{x})]^2}. \tag{17}$$

As in eqn (12), the distribution of dislocations, $f(t)$, is separated into a singular and into a regular part, $F(t)$, resulting in:

$$f(t) = \frac{F(\bar{t})}{\sqrt{R}}\sqrt{\left(\frac{1-\bar{t}}{1+\bar{t}}\right)}. \tag{18}$$

By means of:

$$h_1(\bar{t}) = \frac{F(\bar{t})}{\sqrt{R}\Delta\alpha^*\Delta T}, \quad w_1(\bar{t}) = \frac{1}{\sqrt{(1-\bar{t}^2)}} \tag{19}$$

or, alternatively:

$$h_2(\bar{t}) = \frac{F(\bar{t})}{(\kappa - R)\Delta\alpha^*\Delta T(1-\bar{t})}, \quad w_2(\bar{t}) = \sqrt{\left(\frac{1-\bar{t}}{1+\bar{t}}\right)} \tag{20}$$

the CSIE (16) can also be expressed as:

$$\int_{-1}^{+1} \frac{h_i(\bar{t})w_i(\bar{t})}{\bar{t}-\bar{x}}\,d\bar{t} + \int_{-1}^{+1} \frac{2Rk(\bar{x},\bar{t})h_i(\bar{t})w_i(\bar{t})}{(1-\bar{x})(1-\bar{t})}\,d\bar{t} = -\frac{2\pi(1+\alpha)}{1+\alpha-2\beta}\frac{1-\bar{x}}{[2+\varepsilon+R(1-\bar{x})]^2}, \quad i = 1, 2. \tag{21}$$

Note that for $i = 1$ the CSIE is of the type $\kappa = 1$ [cf. for example, Erdogan *et al.* (1973) or Golberg (1990)] which is the type of CSIEs considered in the papers by Erdogan and co-workers (1974, 1975) and more recently by Müller and Schmauder (1993) on cracks of finite size in the vicinity of inclusions. However, for the choice $i = 2$ the CSIE is of the type $\kappa = 0$. Consequently, in each case it becomes necessary to apply the appropriate numerical technique to obtain a solution. This will be discussed in more detail in Section 5.

For the time being it should be emphasized that an additional condition is required in order to guarantee a unique solution of the integral equation (21) for the choice $i = 1$: as shown in eqn (2), the stresses around a thermally mismatched inclusion decrease as $1/x^2$. This will lead to a constant crack opening displacement at infinity (Kemmer, 1994). Thus the gradient of vertical displacement, v, in the horizontal direction, x, must vanish at infinity which, in turn, is linked to the dislocation density, $f(x)$, as follows:

$$f(x) = \frac{\partial}{\partial x}[v(x, +0) - v(x, -0)]. \tag{22}$$

In order to meet this requirement and referring to eqns (18) and (19), it is certainly sufficient to impose the so-called Kutta condition [cf. Golberg (1990)] which in the terminology of this paper reads:

$$h_1(+1) = 0. \tag{23}$$

Moreover, note that the Kutta condition (23) has been used before, e.g. by Ming-Che Lu and Erdogan (1983), to compute SIFs for an edge crack; and indeed, this is no coincidence since, suggestively speaking, the semi-infinite crack is an edge crack with its edge at infinity.

3. STRESS INTENSITY FACTORS

The SIF at the tip of a semi-infinite crack can be computed as follows [see, for example, Erdogan *et al.* (1974); Erdogan and Gupta (1975); Erdogan (1983); Tang and Erdogan (1984); Müller and Schmauder, (1993)]:

$$K_I(R+a) = \frac{2\mu_1}{1+\kappa_1}\sqrt{(2\pi)}\lim_{x\to R}[(x-R)+a]^{1/2}f(x). \tag{24}$$

Inserting eqns (12) and (13) into this expression yields:

$$\frac{K_I}{K_0} = \sqrt{(2)}h_1(0), \quad K_0 = \frac{2\mu_1}{1+\kappa_1}\Delta\alpha^*\Delta T\sqrt{(\pi R)}. \tag{25}$$

Alternatively, by using eqns (18), (20) the following results are obtained:

$$\frac{K_I}{K_0} = \sqrt{(2)}h_1(-1), \quad \frac{K_I}{K_0} = 2^{1/2}h_2(-1). \tag{26}$$

4. A SELF-CONSISTENT FORMULA

An approximate analytical K-solution for the semi-infinite crack in the neighborhood of a thermally mismatched inclusion (cf. Fig. 1) can be obtained by employing a self-consistent procedure. It is first assumed that the distance between the crack tip and the center of the inclusion is much larger than the radius of the inclusion. As a result, in the vicinity of the crack tip, one will not see the detailed shape of the inclusion, and the dissimilar inclusion can be treated as a spot with effective transformation strains. Similarly, in the neighborhood of the inclusion, one will not see the location of the crack tip, and the influence of the crack on the inclusion can approximately be represented by an effective homogeneous loading. The relations among the effective loading, the stress intensity factor and the effective transformation strains lead to a self-consistent loop: the effective loading depends on the SIF at the crack tip, the SIF is determined by the transformation strains, and the transformation strains are related to the effective loading through an Eshelby-type formula. Solving these loop relations yields [cf. Kachanov *et al.* (1990) for an analogous treatment of crack–microcrack interaction, Rodin and Yuh-Long Hwang (1991) for the case of inclusion-inclusion interaction and Gong and Meguid (1992) for crack interaction with an elliptical hole]:

$$\frac{K_I}{K_0} = \frac{1+\alpha}{1+\alpha-2\beta}\frac{\sqrt{2}}{(1+\varepsilon/R)^{3/2}}. \tag{27}$$

Although the self-consistent formula is strictly valid only when the crack tip is far from the inclusion, it will be shown that it gives reasonable predictions even if the crack tip is located in the close neighborhood of the inclusion.

In the self-consistent procedure the inclusion is treated as a transformed spot. On the other hand, a moduli-perturbation approach presented by Gao (1991) can be used to calculate the effect of the inclusion shape at a close distance, but it will be accurate only for small differences between the elastic constants of the matrix and the inclusion. In principle, a more superior solution can be constructed by asymptotically interpolating between the self-consistent solution and the moduli-perturbation solution. This, however, is left to future work.

The special case of a radial semi-infinite crack in front of a circular inclusion with the *same* elastic constants as the matrix can alternatively be obtained from the corresponding *exact* analytical solution for the finite crack which was derived in a paper by Müller (1990) by solving the integral equation (8). This was achieved by using an inversion formula as provided by Muskhelishvili (1992) in Section 90 of his book on singular integral equations. The formula for the finite crack reads:

$$\frac{K_I}{K_0} = 2\left(\frac{R}{l}\right)^{3/2}\frac{L/l+1}{(L^2/l^2-1)^{3/2}}, \tag{28}$$

where L denotes the distance from the center of the particle to the center of the crack and $2l$ is the total crack length. Hence it follows that:

$$\frac{L}{l} = 1+\frac{\varepsilon+R}{l}. \tag{29}$$

In the limit of a semi-infinite crack ($l \to \infty$) eqn (28) together with eqn (29) lead to eqn (27), when specialized to the case $\alpha = \beta = 0$.

Note that the (α, β)-factor in the analytical K-solution shown in eqn (27) also appears on the right-hand side of the integral equations (14) or (21). This is by no means just a coincidence. In fact, looking at the solution of the CSIE for the homogeneous case and by interpreting and ignoring the kernel, $k(x, t)$, in the original integral equation (21), as a second order perturbation [cf. Golberg, (1990)], yields:

$$\int_{-1}^{1} \frac{h(\bar{t})w(\bar{t})}{\bar{t}-\bar{x}}\,d\bar{t} = -\frac{2\pi(1+\alpha)}{1+\alpha-2\beta}\frac{1-\bar{x}}{[2+\varepsilon/R(1-\bar{x})]^2}. \tag{30}$$

This equation can now be solved analytically by means of the same techniques as used by Müller (1990). The result is eqn (27) which describes the case of slight elastic mismatch. Thus it can be said that, within first order, the influence of elastic mismatch on the stress intensity factor results from the stresses around the mismatched particle in the undamaged matrix, i.e. $p(x)$. The interaction between the crack and the particle, i.e. the kernel, $k(x,t)$, will only produce higher order terms.

5. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

In order to solve the CSIEs (14) or (21) for arbitrary choices of Dundurs' parameters, α and β, numerical procedures are needed. In the case of CSIEs with an infinite boundary a possible method is the Radau–Chebyshev discretization scheme which is described in a paper by Ioakimidis and Theocaris (1980) and which reads as follows:

$$\sum_{i=1}^{N} W_i h(\bar{t}_i)\left\{\frac{1}{\bar{t}_i-\bar{x}_k}+Rk(\bar{x}_k,\bar{t}_i)\right\} = -\frac{2\pi(1+\alpha)}{1+\alpha-2\beta}\frac{1}{[1+\bar{x}_k+\varepsilon/R]^2}$$

$$\bar{t}_i = \frac{1}{c}\ln\frac{2}{1-t_i}, \quad t_i = \cos\left(\frac{i\pi}{N}\right),$$

$$\bar{x}_k = \frac{1}{c}\ln\frac{2}{1-x_k}, \quad x_k = \cos\left(\frac{\pi[k-0.5]}{N}\right), \quad i,k = 1,\ldots,N$$

$$W_i = \frac{A_i}{2c}\exp\left(\frac{3c\bar{t}_i}{2}\right)\left(\frac{1-\exp[-c\bar{t}_i]}{\bar{t}_i}\right)^{1/2}, \quad i = 1,\ldots,N$$

$$A_i = \frac{\pi}{N}(1-t_i), i = 1,\ldots,N-1, A_N = \frac{\pi}{N}. \tag{31}$$

where c is an arbitrary constant which, based on the results of the aforementioned paper by Ioakimidis and Theocaris, was chosen to be 0.1 in the following computations.

Alternatively to eqns (31), the CSIE (14) could, in principle, be solved by using the Gauss–Hermite integration technique [cf. Ioakimidis (1976), Section Δ9 or Ioakimidis and Theocaris (1977)]. In this case the weights W_i are given by:

$$W_i = A_i\frac{\exp(x_i)(2N+1)!\sqrt{\pi}}{(2N+1)^2(N!)^2 2^{2N}(L_{N-1}^{1/2}[t_i])^2}, \quad i = 1,\ldots,N$$

$$A_i = 2, \quad i = 1,\ldots,N-1, \quad A_N = 1. \tag{32}$$

where the symbol $L_{N-1}^{1/2}$ stands for a certain Laguerre polynomial, t_i are the zeros of the following Hermite polynomial:

$$H_{2N+1}(t) = 0, \quad i = 2,\ldots,N, \quad t_1 = 0 \tag{33}$$

and the x_i are the zeros of Kummer's confluent hypergeometric function:

$${}_1F_1(-N;N+1;x_k) = 0, \quad k = 1,\ldots,N. \tag{34}$$

For the case $N = 5$ these formulae were evaluated in the two aforementioned references.†

† Note that the value 13.541 in the table on p. 373 of the thesis work by Ioakimidis (1976) is erroneous and needs to be replaced by 14.29787 as quoted in the paper by Ioakimidis and Theocaris (1977).

However, it should be pointed out that choosing $N = 5$ is insufficient to obtain accurate results for cracks in the close vicinity of a thermally mismatched inclusion and that the Gauss–Hermite integration technique becomes unnecessarily cumbersome for higher values of N. Therefore it will not be used in the following numerical analyses and the much simpler Radau–Chebyshev method shown in eqns (31) will be applied instead.

Thus, the stress intensity factor of eqn (25) can be approximated by:

$$\frac{K_I}{K_{I0}} \approx (\sqrt{2})h(\bar{t}_N). \tag{35}$$

For the numerical study of the CSIE (21) of index $\kappa = 1$, i.e. for $i = 1$, a Gauss–Chebyshev discretization scheme can be used [cf. Erdogan *et al.* (1973)] which leads to the following set of linear equations:

$$\sum_{i=1}^{N} W_i h(\bar{t}_i)\left\{\frac{1}{\bar{t}_i - \bar{x}_k} + \frac{2Rk(\bar{x}_k, \bar{t}_i)}{(1-\bar{x}_k)(1-\bar{t}_i)}\right\} = -\frac{2\pi(1+\alpha)}{1+\alpha-2\beta}\frac{1-\bar{x}_k}{[2+\varepsilon/R(1-\bar{x}_k)]^2}$$

$$h(\bar{t}_1) = 0, \quad W_i = \frac{\pi}{N},$$

$$\bar{t}_i = \cos\left(\pi\frac{2i-1}{2N}\right), \quad i = 1, \ldots, N, \quad \bar{x}_k = \cos\left(\frac{\pi k}{N}\right), \quad k = 1, \ldots, N-1. \tag{36}$$

Note that it is possible to solve this set of linear equations only because of the Kutta condition introduced in eqn (23).

Moreover, the SIF of eqn (26) can be approximated by:

$$\frac{K_I}{K_{I0}} \approx (\sqrt{2})h_1(\bar{t}_N). \tag{37}$$

If convergence implied by the set of linear eqns (36) becomes an issue (as it will if the crack tip is extremely close to the thermally and elastically mismatched fiber) it is advisable to interpolate the discrete solution data $h_1(\bar{t}_i)$, $i = 1, \ldots, N$, e.g. by using a polynomial expression which can be evaluated at $t = -1$. Using Lagrange's interpolation formula [cf. for example, Hildebrand (1974)] it can be shown that a possible polynomial $h_1^{N-1}(\bar{t})$ of order $N-1$ is given by the following equation:

$$h_1^{N-1}(\bar{t}) = \frac{T_N(\bar{t})}{N}\sum_{i=1}^{N}\frac{h_1(\bar{t}_i)\sqrt{(1-\bar{t}_i^2)}}{(\bar{t}-\bar{t}_i)(-1)^{i+1}}, \tag{38}$$

where $T_N(\bar{t})$ is the Nth Chebyshev polynomial of the first kind. If this polynomial is evaluated at point $\bar{t} = -1$ the following compact approximate expression for $h_1(\bar{t})$ results:

$$h_1^{N-1}(-1) = \frac{1}{N}\sum_{i=1}^{N} h_1(\bar{t}_i)(-1)^{N-i}\sqrt{\left(\frac{1-\bar{t}_i}{1+\bar{t}_i}\right)}. \tag{39}$$

Alternatively, it is possible to use an algorithm which takes the crack tips, i.e. the end points -1 and $+1$, *a priori* into account. This algorithm is known as Lobatto–Chebyshev discretization and it has been studied in great depth in the thesis work of Ioakimidis (1976, Section Γ3) and also, for example, in a paper by Theocaris and Ioakimidis (1977):

$$h(\bar{t}_1) = 0, \quad W_i = \frac{\pi \lambda_i}{N-1}, \quad \lambda_i = 1, \quad \lambda_1 = \lambda_N = 0.5$$

$$\bar{t}_i = \cos\left(\pi \frac{i-1}{N-1}\right), \quad i = 2, \ldots, N-1, \quad \bar{t}_1 = +1, \quad \bar{t}_N = -1$$

$$\bar{x}_k = \cos\left(\pi \frac{2k-1}{2(N-1)}\right), \quad k = 1, \ldots, N-1. \tag{40}$$

Turning now to the case of CSIEs of index $\kappa = 0$, i.e. for $i = 2$, the following Gauss–Jacobi discretization scheme should be used [cf. Erdogan *et al.* (1973)]:

$$W_i = \frac{2(1-\bar{t}_i)}{2N+1},$$

$$\bar{t}_i = \cos\left(\pi \frac{2i}{2N+1}\right), \quad \bar{x}_k = \cos\left(\pi \frac{2k-1}{2N+1}\right), \quad i,k = 1, \ldots, N. \tag{41}$$

Obviously, no further conditions are required for a solution of the resulting set of linear equations and the stress intensity factor of eqn (26) reads:

$$\frac{K_I}{K_0} \approx 2^{3/2} h_2(\bar{t}_N). \tag{42}$$

In analogy to eqn (38) it is possible to compute a Lagrangian interpolation polynomial $h_2^{N-1}(\bar{t})$ of order $N-1$ as follows:

$$h_2^{N-1}(\bar{t}) = \frac{P_N^{(1/2,-1/2)}(\bar{t})}{N} \sum_{i=1}^{N} \frac{h_2(\bar{t}_i)(1-\bar{t}_i^2)}{(\bar{t}-\bar{t}_i)P_N'^{(1/2,-1/2)}(\bar{t}_i)}, \tag{43}$$

where $P_N^{(1/2,-1/2)}$ and $P_N'^{(1/2,-1/2)}(\bar{t})$ are the Nth Jacobi polynomial of index $(\frac{1}{2}, -\frac{1}{2})$ and its derivative, respectively, and $h_2(\bar{t}_i)$, $i = 1, \ldots, N$ denotes discrete solution data. If this polynomial is evaluated at point $\bar{t} = -1$ the following approximate expression for $h_2(\bar{t})$ can be obtained:

$$h_2^{N-1}(-1) = \frac{(N-2)(-1)^{N-1}}{2N+1} \sum_{i=1}^{N} \frac{h_2(\bar{t}_i)\sin\left(\pi \frac{i}{2N+1}\right)}{\sin\left(\pi \frac{2Ni}{2N+1}\right)} \frac{1-\bar{t}_i}{\sqrt{(1+\bar{t}_i)}}. \tag{44}$$

6. RESULTS AND DISCUSSION

The sequence of three-dimensional plots in Fig. 2 presents SIFs for a semi-infinite crack in front of a thermally mismatched, circular inclusion at different distances ε/R for the full range of Dundurs' parameters possible [cf. Suga *et al.* (1988)]. The SIFs were computed both ways, using the analytical formula shown in eqn (27) as well as numerically with eqn (37) by solving the set of linear equations presented in eqns (36). The number of integration points chosen for each calculation was $N = 200$ which, based on prior experience [cf. Müller and Schmauder, (1993)], guarantees high accuracy (error $\ll 1\%$) at least as long as the crack tip is not closer to the inclusion than $\varepsilon/R = 0.0001$. Note that the spikes in the back of all plots are an artifact due to the discreteness of the computed SIF data and to the incapability of the plotting routine to interpolate in a direction not perpendicular either to the α or to the β axis.

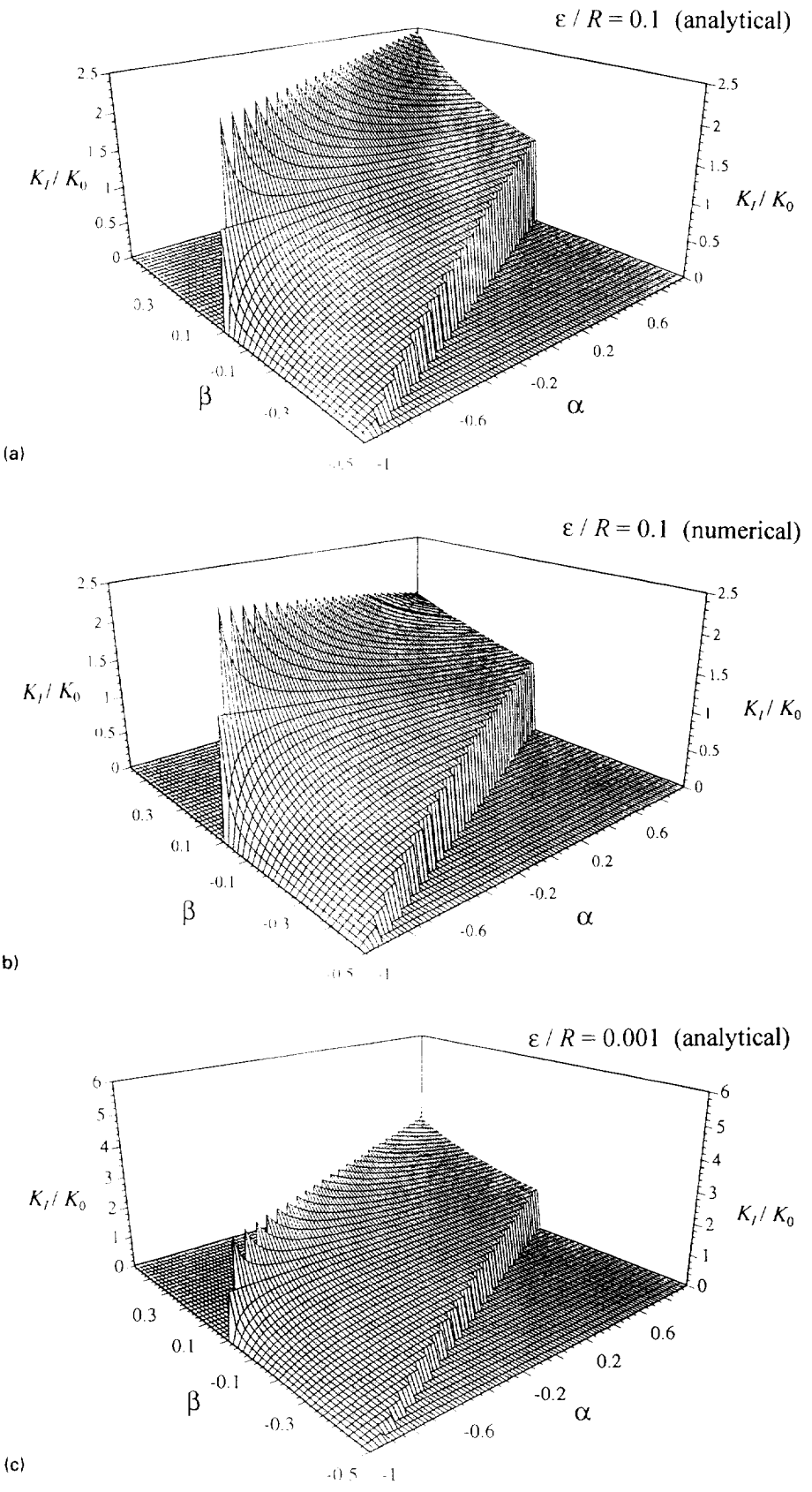
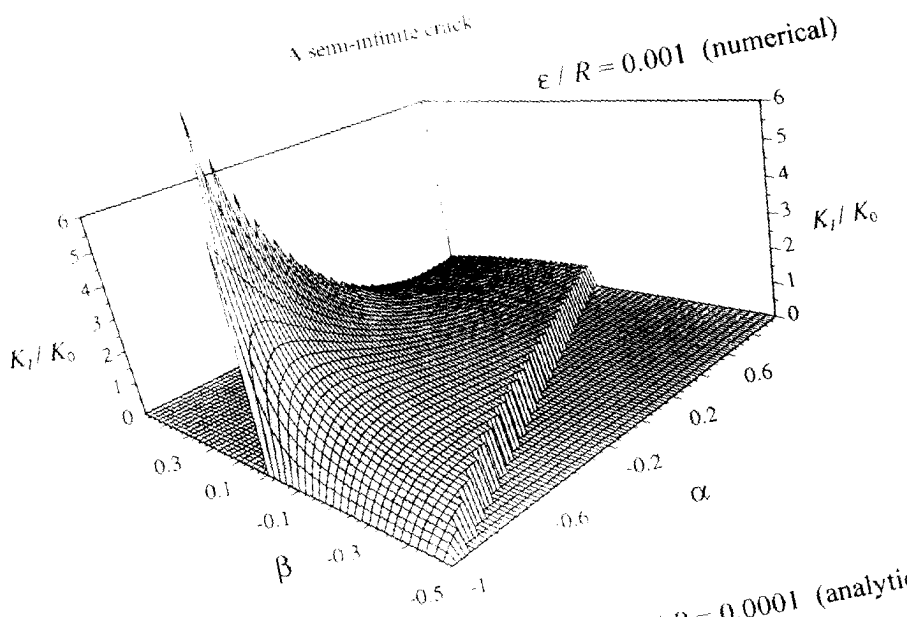
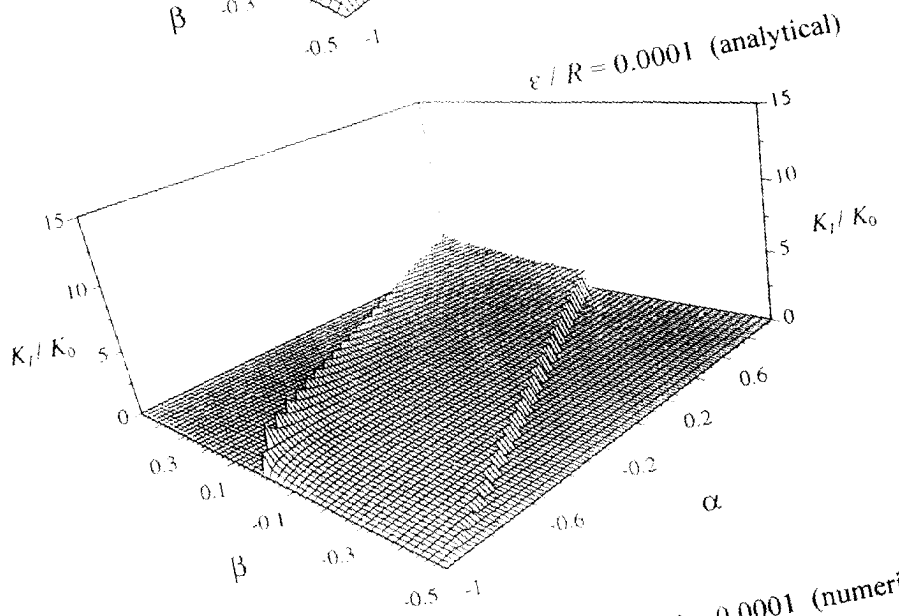


Fig. 2. Analytically and numerically computed SIFs as a function of Dundurs parameters for various distances ε / R.

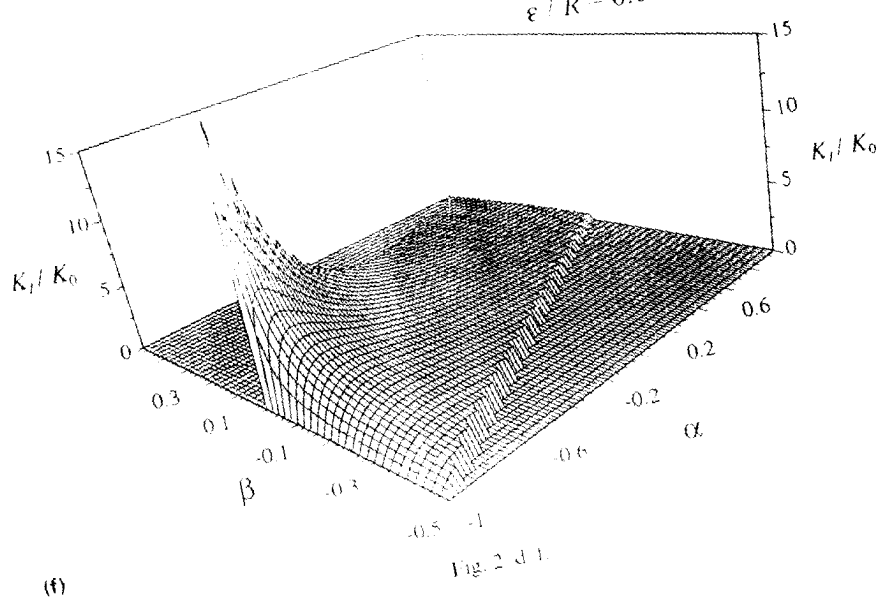
A semi-infinite crack



(d)



(e)



(f)

Fig. 2. d, f.

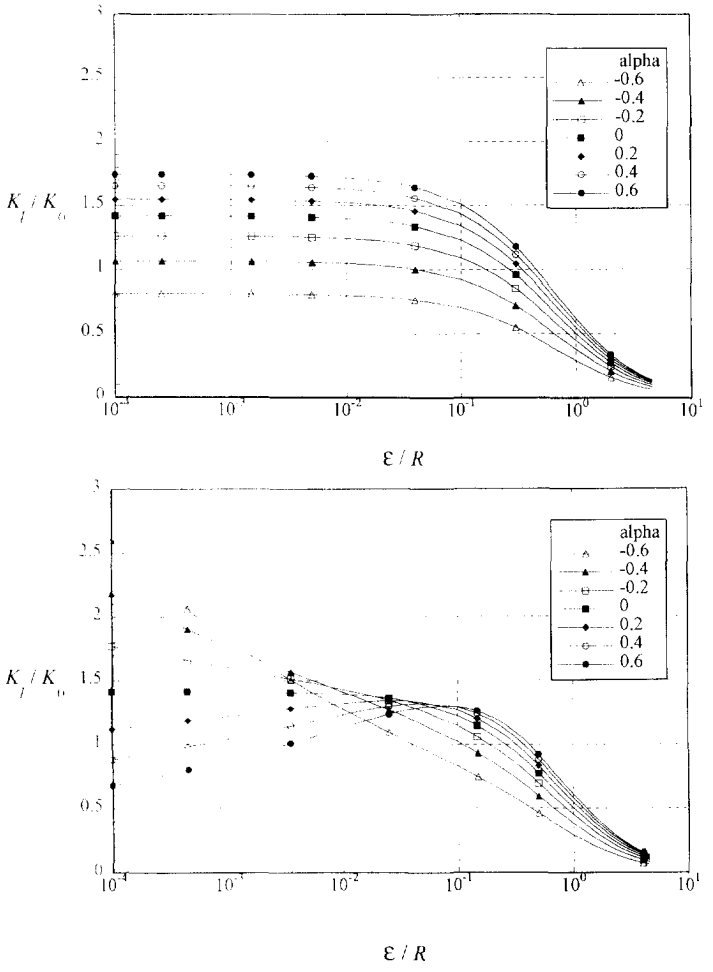


Fig. 3. Analytically and numerically computed SIFs as a function of normalized distance for different Dundurs parameters.

If the crack tip is comparatively far away from the inclusion ($\varepsilon/R \geqslant 0.1$), the analytical and numerical procedures in general lead to very similar results for all choices of Dundurs' parameters. However, for large positive values of α and β the analytical formula (27) tends to overestimate the SIFs by 60%. This is, of course, no surprise since eqn (27) is only a first order approximation with respect to the Dundurs' parameters. Note that the same effect does not occur at large negative values of α and β, which must be attributed to the rapid decrease of the thermal stress field if $\alpha \to -1$ [see eqn (2)].

If the crack tip moves closer and closer towards the inclusion ($\varepsilon/R = 0.001$ or 0.0001) the discrepancies between the analytical and numerical predictions of SIFs become more and more pronounced. In particular, the analytical formula (27) does not account for the fact that negative Dundurs' parameter α and positive Dundurs' parameter† β lead to SIFs which increase rapidly if the crack tip approaches the inclusion. This behavior has been observed before by Ming-Che Lu and Erdogan (1983) for cracks of finite size.

The two graphs presented in Fig. 3 allow the behavior of the SIFs at very small distances to be examined more closely: they show K_I/K_0 as a function of ε/R on a logarithmic scale from values as small as 10^{-4} up to 10^{1}. For convenience only the first Dundurs' parameter, α, has been changed while the second one, β, has been chosen to be:

$$\beta = \frac{\alpha}{4} \tag{45}$$

which is an upper bound for many material combinations [cf. Suga *et al.* (1988)]. As it was

† In this case the crack runs into an inclusion which is softer than the matrix.

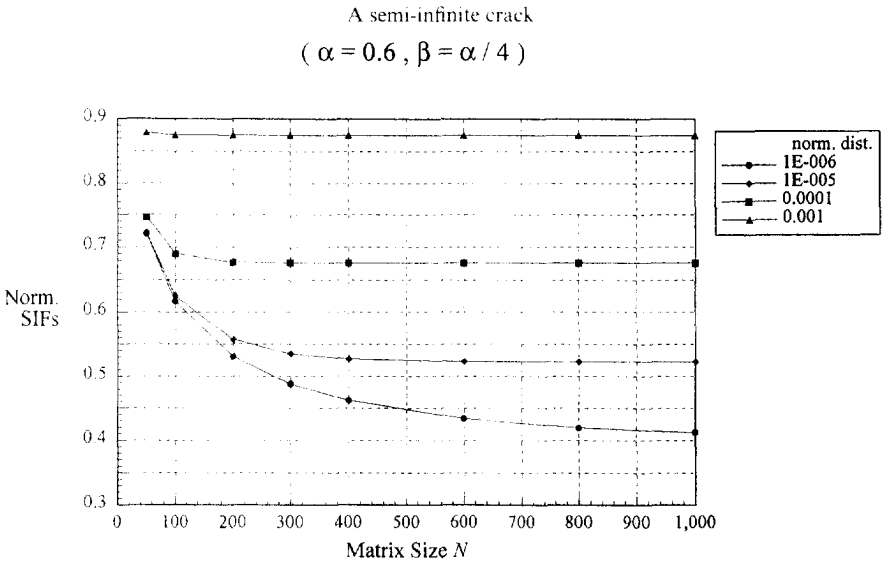


Fig. 4. A test for numerical accuracy of the Gauss–Chebyshev procedure.

observed before, the SIF values predicted by the analytical formula (27) agree very well with the numerical results at distances $\varepsilon/R \geqslant 0.1$. However, for smaller ε/R they quickly converge to [cf. eqn (29)]:

$$\left.\frac{K_I}{K_0}\right|_{\varepsilon/R \to 0} = \frac{(\chi - 2)(1+\alpha)}{1-2\beta+\alpha}. \tag{46}$$

The numerically computed SIFs behave quite differently for small values of ε/R. For positive Dundurs' parameters, the normalized SIFs run through a maximum between $\varepsilon/R = 0.01$ and $\varepsilon/R = 0.1$ to decrease further if $\varepsilon/R \to 0$. For negative Dundurs' parameters the normalized SIFs increase with decreasing ε/R. However, if $\alpha = -1$,† the SIFs are equal to zero because of the vanishing thermal stress shown in eqn (2).

It should be noted that for $\varepsilon/R < 10^{-4}$ it is not sufficient to choose $N = 200$ in eqns (36) in order to obtain a reliable result. This is illustrated in Fig. 4 which presents SIFs normalized according to eqn (25) as a function of matrix size N for a special choice of Dundurs' parameters, α and β. It is clearly visible that the SIFs will converge only weakly if $\varepsilon/R = 10^{-5}$ or 10^{-6}. The same remark holds for other conventional types of discretization, such as the Radau–Chebyshev discretization [Fig. 5, eqns (31)] or the Gauss–Jacobi procedure [Fig. 6, eqns (41)]. To achieve faster convergence it is imperative to use either the polynomial forms in eqns (38) or (43) and to extrapolate into the crack tip [eqns (39) or (44), cf. Figs 7 and 8], or to start with an algorithm which, *a priori*, takes the crack points into account: eqns (40) and Fig. 9. The beneficial influence of such an approach is clearly visible.

7. CONCLUSIONS AND OUTLOOK

The behavior of SIFs of a semi-infinite crack in front of a thermally and elastically mismatched, circular inclusion has been studied for the full range of Dundurs' parameters. The SIFs were computed analytically using a formula, based on a self-consistent method or on a CSIE inversion formula, as well as numerically using discretization methods developed by Erdogan *et al.* and Theocaris and Ioakimidis.

† In this case the inclusion degenerates into a hole.

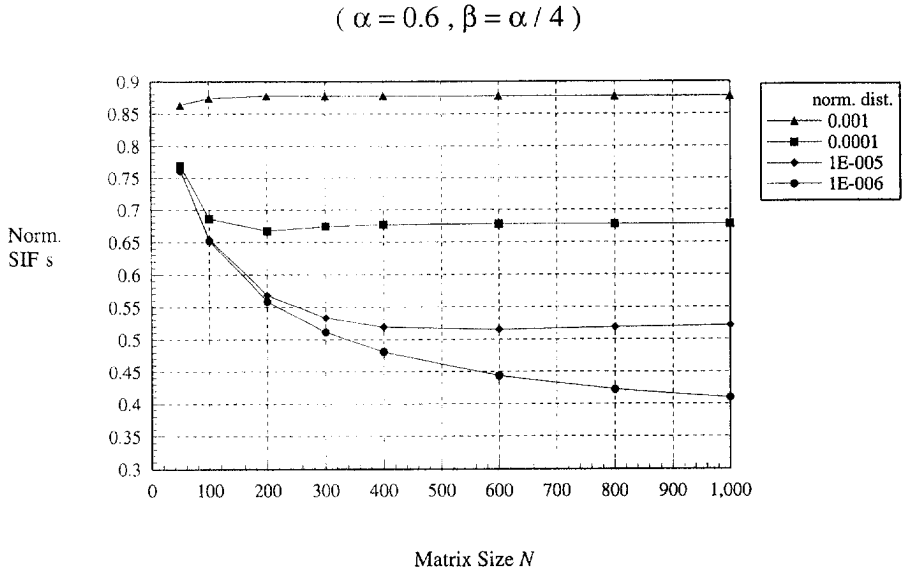


Fig. 5. A test for numerical accuracy of the Radau–Chebyshev procedure.

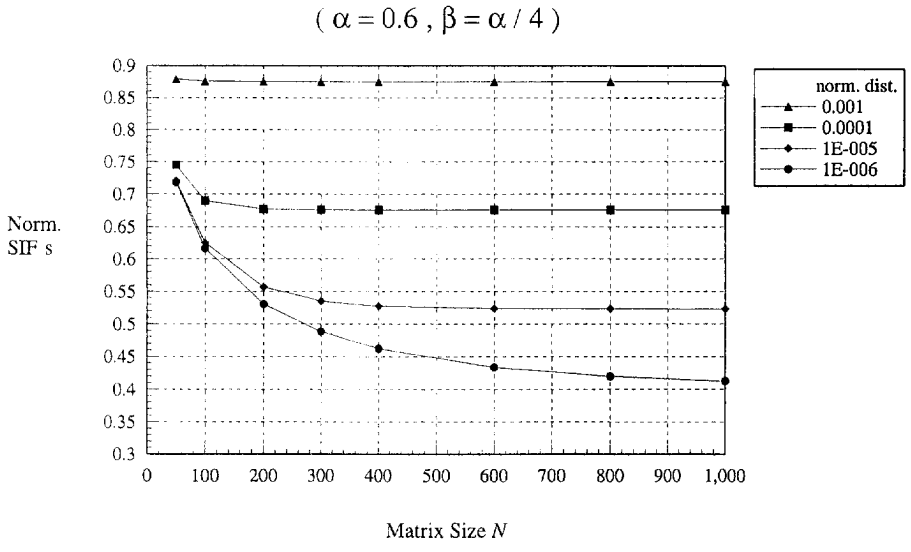


Fig. 6. A test for numerical accuracy of the Gauss–Jacobi procedure.

For small and medium values of Dundurs' parameters the agreement between analytical and numerical procedures is excellent. However, for large, positive Dundurs' parameters the SIFs are overestimated by the analytical formula. This is not surprising since the equation was derived as a first order approximation and is *a priori* valid only for a small elastic mismatch.

Moreover, the validity of this formula is restricted to distances $\varepsilon/R \geqslant 0.1$ where ε denotes the distance of the crack tip from the particle surface and R is the radius of the inclusion. This may be sufficient for all practical purposes, however it should be noted that, depending upon the Dundurs' parameters, SIFs will either start to decrease or increase if $\varepsilon/R \to 0$. Note that the deviation between the SIF predicted by the self-consistent formula and the corresponding numerical result is less than 7% if $|\alpha| \leqslant 0.6$, $\beta = \alpha/4$ and $\varepsilon/R \geqslant 0.2$. Moreover, note that for crack tips extremely close at the heterogeneity ($\varepsilon/R < 10^{-6}$) it is imperative to extrapolate into the crack tips by polynomial interpolation of the discretized

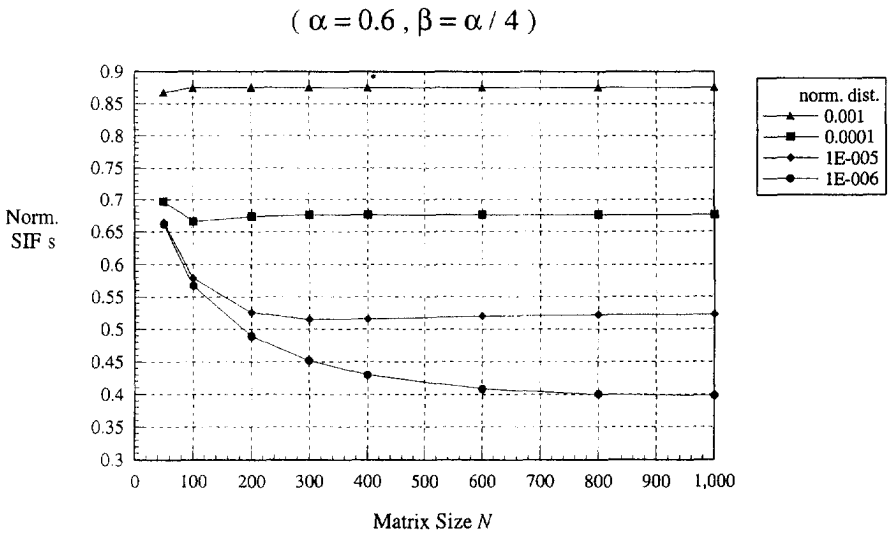


Fig. 7. A test for numerical accuracy of the Gauss–Chebyshev procedure with polynomial extrapolation.

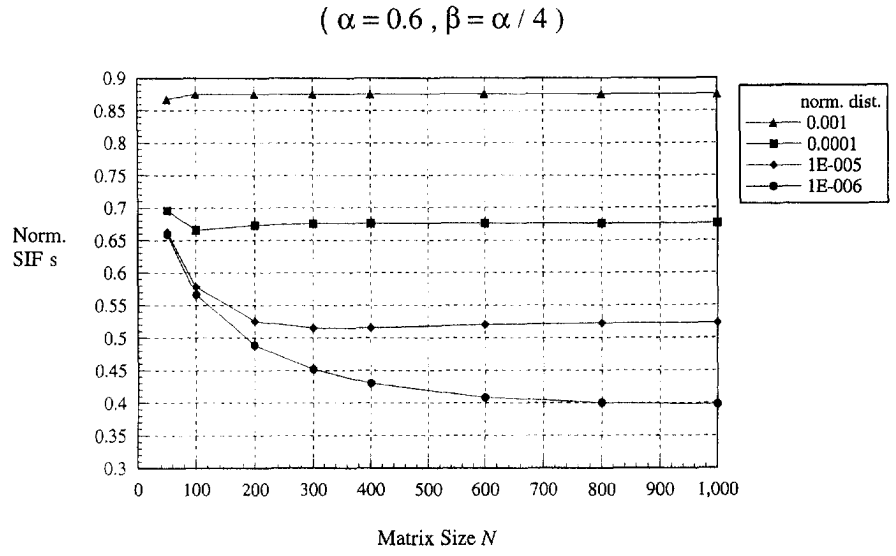


Fig. 8. A test for numerical accuracy of the Gauss–Jacobi procedure with polynomial extrapolation.

solution data or to use integration schemes which take the crack tips explicitly into account (e.g. Lobatto–Chebyshev). Otherwise the numerical effort to obtain accurate results becomes unbearably large: $N \gg 1000$.

Finally, it should be noted that the analytical as well as numerical techniques can both be easily applied to other load and geometry configurations for the semi-infinite crack. Papers dealing with the partially pressurized semi-infinite crack in front of a bimaterial wall or a semi-infinite crack under arbitrary loading conditions, close to a bimaterial interface, are in preparation.

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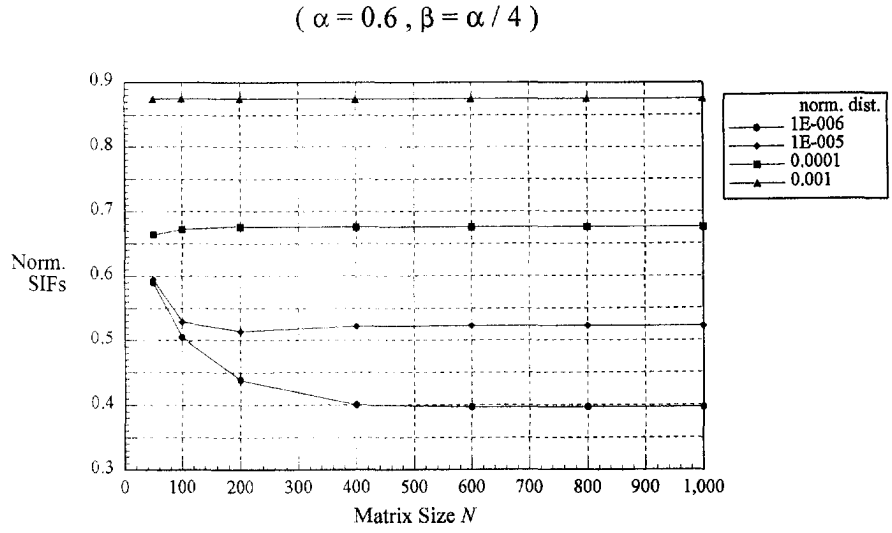

Fig. 9. A test for numerical accuracy of the Lobatto–Chebyshev procedure.

REFERENCES

Erdogan, F. (1983). Stress intensity factors. *J. Appl. Mech.* **50**, 992–1002.

Erdogan, F., Gupta, G. D. and Cook, T. S. (1973). Numerical solution of singular integral equations. In *Mechanics of Fracture* 1: *Methods of Analysis and Solution of Crack Problems* (Edited by G. C. Sih), pp. 369–425. Noordhoff, Leyden.

Erdogan, F. and Gupta, G. D. (1975). The inclusion problem with a crack crossing the boundary. *Int. J. Fracture* **11**, 13–27.

Erdogan, F., Gupta, G. D. and Ratwani, M. (1974). Interaction between a circular inclusion and an arbitrarily oriented crack. *J. Appl. Mech.* **41**, 1007–1013.

Gao, H. (1991). Fracture analysis of nonhomogeneous materials via a moduli-perturbation technique. *Int. J. Solids Structures* **27**, 1663–1682.

Golberg, M. A. (1990). Introduction to the numerical solution of Cauchy singular integral equations. In *Numerical Solution of Integral Equations* (Edited by M. A. Golberg), pp. 183–308. Plenum Press, New York and London.

Gong, S. X. and Meguid, S. A. (1992). Microdefect interacting with a main crack: a general treatment. *Int. J. Mech. Sci.* **34**, 933–945.

Hildebrand, F. B. (1974). Introduction to numerical analysis. In *International Series in Pure and Applied Mathematics*, 2nd Edn. McGraw-Hill, New York.

Hutchinson, J. W., Mear, M. E. and Rice, J. R. (1987). Crack paralleling an interface between dissimilar materials. *J. Appl. Mech.* **54**, 828–832.

Ioakimidis, N. I. (1976). General methods for the solution of crack problems in the theory of plane elasticity (in Greek). Thesis, National Technical University of Athens (Greece), Xerox University Microfilms, Ann Arbor, Michigan.

Ioakimidis, N. I. and Theocaris, P. S. (1977). The Gauss–Hermite numerical integration method for the solution of the plane elastic problem of semi-infinite periodic cracks. *Int. J. Engng Sci.* **15**, 271–280.

Ioakimidis, N. I. and Theocaris, P. S. (1980). The practical evaluation of stress intensity factors at semi-infinite crack tips. *Engng Fracture Mech.* **13**, 31–42.

Kachanov, M., Montagut, E. L. E. and Laures, J. P. (1990). Mechanics of crack–microcrack interactions. *Mech. Mater.* **10**, 59–71.

Kemmer, G. (1994). Research Report, Universität Paderborn.

Ming-Che Lu and Erdogan, F. (1983). Stress intensity factors in two bonded elastic layers containing cracks perpendicular to and on the interface—II. Solution and results. *Engng Fracture Mech.* **18**, 507–528.

Müller, W. H. (1990). An analytical approach for the calculation of stress-intensity factors in transformation-toughened ceramics. *Cont. Mech. Thermodyn.* **2**, 279–299.

Müller, W. H. and Schmauder, S. (1993). Stress-intensity factors of *r*-cracks in fiber-reinforced composites under thermal and mechanical loading. *Int. J. Fracture* **59**, 307–343.

Muskhelishvili, N. I. (1992). *Singular Integral Equations*, 2nd edn. Dover, New York.

Rodin, G. J. and Yuh-Long Hwang (1991). On the problem of linear elasticity for an infinite region containing a finite number of non-intersecting spherical inhomogeneities. *Int. J. Solids Structures* **27**, 145–149.

Suga, T., Elssner, G. and Schmauder, S. (1988). Composite parameters and mechanical compatibility of material joints. *J. Compos. Mater.* **22**, 917–934.

Tang, R. and Erdogan, F. (1984). Stress intensity factors in a reinforced thick-walled cylinder. *Int. J. Engng Sci.* **22**, 867–879.

Theocaris, P. S. and Ioakimidis, N. I. (1977). Numerical integration methods for the solution of singular integral equations. *Q. Appl. Math.* **35**, 173–183.